AGT conjecture, perverse sheaves on instanton moduli and what I learned this week.

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Knot homologies and BPS states

AGT conjecture

Nekrasov deformed partition function for $\mathcal{N}=2$ SUSY (pure) YM theory on $\mathbb{R}^4$

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = (v|v)$$

v: Whittaker vector in Verma module of the W-algebra

mathematical formulation

G : compact Lie group

$M(G,n)$ = moduli space of G-instantons on $S^4 = \mathbb{R}^4 \cup \infty$ $(\mathbb{R}^4 = \mathbb{C}^2)$ with framing at ∞, "instanton number" $= n$

$$\circlearrowleft \mathbb{T} := \underbrace{U(1) \times U(1)}_{\text{base}} \times \underbrace{T}_{\text{framing}} \subset U(2) \times G$$

Conjecture. $\bigoplus_n H^*_{\mathbb{T}}(M(G,n))$ has a structure of the (dual) Verma module of the W-algebra $W(\hat{\mathfrak{g}}^\vee_{\mathbb{C}})$

equivariant cohomology, precise later

$c, \Delta \leftarrow$ equivariant parameters

such that $\sum$ [the fundamental classes] $= v$

Conjecture is proved when $G = SU(N)$ (or more precisely $U(N)$)
by $\begin{cases} \text{Maulik–Okounkov} \\ \text{Schiffmann–Vasserot} \end{cases}$

Today : Give a formulation working for general G
$\longrightarrow$ a step towards a proof for general G

Key Use IC sheaves on moduli spaces of instantons.
objects in an abelian category $\mathcal{A}$ (or a derived category)

Goal

$$\boxed{\text{Conj. } U(W(\hat{\mathfrak{g}}^\vee_{\mathbb{C}})) \cong \mathrm{Ext}^\bullet_{\mathcal{A}}(IC, IC) \quad \text{Yoneda product}}$$

Advantage One can use various functors $\mathcal{A} \xrightarrow{f_*} \mathcal{B}$.

Remarks ① Moduli spaces themselves are not fundamental.
IC's are more important
$\mathcal{A}$ may not be fundamental, as we may replace it

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f_*} & \mathcal{B} \\ \Downarrow & & \Downarrow \\ IC & & f_* IC \end{array}$$

Hope : $\mathrm{Ext}^\bullet_{\mathcal{A}}(IC, IC) \sim \mathrm{Ext}^\bullet_{\mathcal{B}}(f_* IC, f_* IC)$ Morita equivalent

This framework nicely fits with the theme of our symposium:

② DT/PT/ ····· $\quad \chi^{\text{Behrend}}$ constructible function on moduli space

$= \text{local Euler char.}(\text{perverse sheaf } L)$

s.t. $\quad \text{DT/PT/}\cdots \text{ invariant} = \int_{\mathcal{M}_{DT/PT/\cdots}} \chi^{\text{Behrend}}$

$= \sum (-1)^i \dim H^i(p_! L) \qquad p : \mathcal{M}_{PT/DT/\cdots} \to pt$

$\text{motivic invariant} = \sum (-1)^i t^j \dim \mathrm{Gr}^j H^i(p_! L)$

$\qquad\qquad \text{weight filtration}$

$\underset{\text{if } p_!L \text{ is pure}}{=} \sum (-t)^i \dim H^i(p_! L)$

It is natural to ask also

What is $\mathrm{Ext}^\bullet_D(L, L)$?

Guess: BPS algebra (Harvey – Moore)

cohomological Hall algebra (Kontsevich – Soibelman)

<u>ex</u> = what I have learned this week : $X = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1) \sim T^*S^3$

- $K \overset{?}{\rightsquigarrow}$ $L = L_K$ perverse sheaf on moduli $\mathcal{M}$ of open DT / PT / $\cdots$ on X
  knot
  s.t. $H^i(p_! L) = H^i_{KhR}(K)$
  $\mathrm{Ext}^\bullet_D(L, L) \ni$ differentials d_N (Sergei's talk)
- $K = T(m,n) \overset{?}{\Longrightarrow}$ $\mathcal{M} =$ Hitchin fibration or $\mathrm{Hilb}^N(\mathbb{C}^2) \to S^N\mathbb{C}$ $(\mathbb{C}^2 \xrightarrow{x^m - y^n} \mathbb{C})$
  [GORS] & $\mathrm{Ext}^\bullet_D(L, L) \overset{?}{\cong} H_n(c = \frac{m}{n})$ rational Cherednick algebra
- K : algebraic knot $\overset{?}{\Longrightarrow}$ L_K should be "algebraic",
  and has a weight filtration.
  $\overset{?}{\longleftrightarrow}$ 4th grading in Mina's talk.

These are more or less what I want to say today.
In the remaining time, I explain why it is natural to consider IC sheaves on moduli spaces.
But they just come from a standard framework in geometric representation theory.

Correction after hearing Yan's talk.

There should exist a universal perverse sheaf L, independent of a knot K.
Then $\mathrm{Ext}^\bullet(L, L)$ is the BPS algebra for closed strings.

A knot K gives another perverse sheaf L_K, and we have

$$\mathrm{Ext}^\bullet(L, L_K) = H^\bullet_{KhR}(K)$$

Then $\mathrm{Ext}^\bullet(L, L)$ acts on $\mathrm{Ext}^\bullet(L, L_K) = H^\bullet_{KhR}(K)$.

The differential d_N comes from $\mathrm{Ext}^\bullet(L, L)$.

Before talking general G, recall a special property of $M(G,n)$ for $G = SU(N)$.

$$\overline{M}(G,n) = \text{Uhlenbeck partial compactification of } M(G,n)$$
$$= M(G,n) \sqcup M(G,n-1)\times\mathbb{R}^4 \sqcup M(G,n-2)\times S^2\mathbb{R}^4 \sqcup \cdots\cdots$$

<u>Fact</u> $\overline{M}(SU(N),n)$ has a semismall resolution of singularities
$$\pi : \widetilde{M}(N,n) \longrightarrow \overline{M}(SU(N),n)$$

where $\widetilde{M}(N,n) =$ moduli space of $\mathrm{rk}=N$ torsion free sheaves on $\mathbb{CP}^2$ with framing at l_∞, $c_2 = n$

$H^*_{\mathbb{T}}(M(G,n))$ in [MO] or [SV] is defined as $H^*_{\mathbb{T}}(\widetilde{M}(N,n))$.

However, $W(\hat{\mathfrak{g}}^\vee_{\mathbb{C}}) = W(\hat{\mathfrak{sl}}_N)$ must be replaced by $W(\hat{\mathfrak{gl}}_N)$.

So we should set
$$H^*_{\mathbb{T}}(M(U(N),n)) \overset{\text{def.}}{=} H^*_{\mathbb{T}}(\widetilde{M}(N,n)).$$

Ex $N=1$ $\quad G=U(1)$ (We should get $W(\mathfrak{g}_{\mathbb{C}}^{\vee})=$ Heisenberg alg.)

Naively $M(U(1),n)=\phi$ unless $n=0$

However $\tilde{M}(1,n)=$ Hilbert scheme of n points in $\mathbb{C}^2$

So natural to define

$$H_{\mathbb{T}}^{*}(M(U(1),n)) \overset{\text{def.}}{=} H_{\mathbb{T}}^{*}(\tilde{M}(1,n))$$

Th (N, Grojnowski 1995)

$\bigoplus_{n} H_{\mathbb{T}}^{*}(M(U(1),n))$ has a structure of the Fock space :

$$[a_i, a_j] = (-1)^{i-1}\, i\, \delta_{i+j,0}$$

This can be considered as the 1^{st} case of AGT.

Return back to $G = SU(N)$

<u>Q.</u> How to cut out Heis. to get $H^*_{\mathbb{T}}(M(SU(N),n))$ from $H^*_{\mathbb{T}}(M(U(N),n)) = H^*_{\mathbb{T}}(\widetilde{M}(N,n))$?

<u>Ans.</u> [BFFR] Use the intersection cohomology $IH^*_{\mathbb{T}}(\overline{M}(SU(N),n))$.

BBDG decomposition thm + π: semismall (BM)

$$\Longrightarrow \quad H^*_{\mathbb{T}}(\widetilde{M}(N,n)) \supset IH^*_{\mathbb{T}}(\overline{M}(SU(N),n))$$

More precisely

$$\mathrm{LHS} \cong \bigoplus_{\substack{n' \leq n \\ |\lambda| = n-n'}} IH^*_{\mathbb{T}}(\overline{M}(SU(N),n') \times S^{n-n'}_{\lambda}\mathbb{R}^4) \otimes H_{\mathrm{top}}(\pi^{-1}(E, C_\lambda))$$

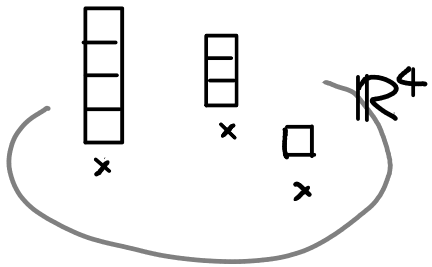
$(E, C_\lambda) \in S^{n-n'}_{\lambda}\mathbb{R}^4$

stratification of Uhlenbeck cptification

1 dim. in our case

$$S^n\mathbb{R}^4 = \bigsqcup_{\lambda \vdash n} S^n_\lambda \mathbb{R}^4$$

configuration of points



BBDG decomposition theorem is better formulated in $D^b_{\mathbb{T}}(\overline{M}(SU(N), n))$

bounded derived category of constructible sheaves

Replace Vector spaces (more precisely $H^*_{\mathbb{T}}(pt)$-modules) $\Longrightarrow$ IC sheaves on moduli spaces

This is a refinement: We can recover

$$H^*_{\mathbb{T}}(\widetilde{M}(N,n)) = H^*(\underbrace{p_! \mathbb{C}_{\widetilde{M}(N,n)}}_{\in D^b_{\mathbb{T}}(pt)})$$

where $\mathbb{C}_{\widetilde{M}(N,n)}$: constant sheaf on $\widetilde{M}(N,n)$

$p : \widetilde{M}(N,n) \longrightarrow pt$

BBDG + BM : $\pi_! \mathbb{C}_{\widetilde{M}(N,n)}[2nN] = \bigoplus IC(M(U(N), n') \times S^n_\lambda \mathbb{R}^4) \otimes H_{top}(\pi^{-1}(E, C_\lambda))$

For general G, there is no analog of $\tilde{M}(N,n)$. So we need to use IC sheaves directly.

AGT conj. in this framework says

$$\boxed{\operatorname{Ext}^{\bullet}_{\oplus D_{\Pi}(\overline{M}(G,n))}\left(\oplus IC(\overline{M}(G,n)), \oplus IC(\overline{M}(G,n))\right) \underset{?}{\cong} U(W(\mathfrak{g}^{\vee}_{\mathrm{aff}}))}$$

<u>Remark</u>. In general, without semismallness assumption, IC's appear with shifts.

IC's — simple objects in the abelian category $\mathrm{Perv}_{\Pi}(\overline{M}(U(N),n)) \subset D^b_{\Pi}(\overline{M}(U(N),n))$

π: semismall $\Longleftrightarrow$ $\pi_!$ preserves Perv.

Technically this is important:
Forgetting Π-action does not loose much information
(refined $\longrightarrow$ unrefined)

As an example of use of this framework,

I construct $\mathcal{W}(\hat{\mathfrak{g}}^\vee_{\mathbb{C}}) \longrightarrow \mathcal{W}(\hat{\mathfrak{l}}^\vee_{\mathbb{C}})$ $\mathfrak{l} = \mathrm{Lie}\, L$

$L \subset G$ Levi subgroup

<u>Technical remark</u>

Suppose $L \cong G_1 \times G_2 \times \cdots \times G_k$ (almost product)

$$\overline{M}(L,n) \neq \coprod_{n_1+\cdots+n_k=n} \overline{M}(G_1,n_1) \times \cdots\cdots \times \overline{M}(G_k, n_k)$$

$$\mathrm{LHS} = \coprod_{n'} M(L,n') \times S^{n-n'}\mathbb{R}^4 = \coprod M(G_1,n_1') \times \cdots \times M(G_k, n_k') \times S^{n-n'}\mathbb{R}^4$$

$$\mathrm{RHS} = \coprod M(G_1,n_1) \times S^{l_1}\mathbb{R}^4 \times \cdots \times M(G_k, n_k) \times S^{l_k}\mathbb{R}^4$$

But $\sigma : \mathrm{RHS} \longrightarrow \mathrm{LHS}$ finite morphism

$\Leftarrow$ easy to handle.

In this case, we should consider $\sigma_! (\mathrm{IC} \text{ on } \mathrm{RHS})$.

I denote it by $\mathrm{IC}(\overline{M}(L,n))$ for brevity.

Choose $\rho : S^1 \to G$ s.t. $L = Z_G(\rho)$ e.g. $\left[\begin{array}{c|c|c} t^{m_1} & & \\ \hline & t^{m_2} & \\ \hline & & \ddots \end{array}\right]$ with block sizes $n_1, n_2, \ldots$

$$\begin{array}{ccc} S^1 & \to & G \\ \cap & & \cap \\ \mathbb{C}^* & \to & G^{\mathbb{C}} \end{array}$$

$$P = \{ g \in G \mid \lim_{t\to 0} \rho(t) g \rho(t)^{-1} \text{ exists} \}$$ parabolic subgroup

<u>Rem.</u> ρ is not uniquely determined by L.
The following construction depends only on P.

$\mathbb{C}^*$ acts on $\overline{M}(G,n)$ through ρ.

We have the following diagram:

$$\overline{M}(G,n) \xhookleftarrow{i} \overline{M}^{P}(G,n) := \{\lim_{t\to 0} \text{ exists}\} \xrightarrow[\lim]{p} \overline{M}(L,n)$$

$$\overline{M}(G,n) \overset{i}{\hookleftarrow} \overline{M}^{P}(G,n) := \{ \lim_{t\to 0} \text{ exists} \} \xrightarrow[\lim]{p} \overline{M}(L,n)$$

<u>Conjecture</u> (True for $G = U(N)$ by [MO])

(1) $p_! i^* : \mathrm{Perv}(\overline{M}(G,n)) \longrightarrow \mathrm{Perv}(\overline{M}(G,n)^{\mathbb{C}^*})$

(2) $\exists$ natural isomorphism

$$p_! i^* (IC(\overline{M}(G,n))) \cong \sigma_* IC(\overline{M}(L,n))$$

(1), (2) correspond to $W_G \to W_L$

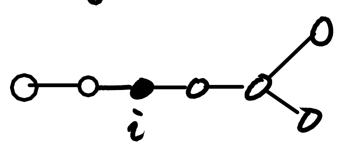
Here "naturality" means the associativity of restriction.

Therefore

$$\begin{tikzcd} \mathrm{Ext}^\bullet \text{ for } G \arrow[rr] \arrow[dr] & & \mathrm{Ext}^\bullet \text{ for } T \cong \bigotimes \mathrm{Heis.} \\ & \mathrm{Ext}^\bullet \text{ for } SU(2)_i \otimes T' \arrow[ur] & \end{tikzcd}$$

$$\mathrm{Ext}^\bullet \text{ for } SU(2)_i \otimes T' \cong \bigotimes \mathrm{Heis} \otimes \mathrm{Vir}_i$$

factors through for each $SU(2)_i \subset G$.



$\Longrightarrow$ $\mathrm{Ext}^\bullet$ for $G = \bigcap \mathrm{Ker}$ of screening operator

$\therefore$ AGT follows.